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Search for Elementary Quantum Statistics

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Bose–Einstein and Fermi–Dirac are the main quantum statistics. Therefore, it is likely that if truly elementary building blocks of Nature exist, they are either bosons or fermions, so that it is also likely that one, and only one, of the following possibilities, concerning those elementary building blocks, is correct: (i) all of them are fermions; (ii) some of them are bosons, others fermions; (iii) all of them are bosons; (iv) the distinction between these cases is methodological, not physical. Since tensors can be constructed from spinors, most physicists support one of the first two points of view. However, by starting from the fact that now it is known that bosonization makes sense, and developing a former research by Penney, we defend the point of view that, at least in a finite model of the Universe, the third point of view is the more likely. To avoid confusion we state that we are not concerned with the whole set of the so-called "elementary particles" since most physicists believe by now that, e.g., hadrons are built from quarks, nor concerned with quarks since many physicists suspect they are also composite objects. This research concerns the true elementary building blocks of Nature, assuming that such set exists, whatever those building blocks are. Finally, we extend this research to general finite associative algebras, enlarging the physical applicability of our point of view concerning the role of bosons in Nature.

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1. PURPOSE OF THIS PAPER

Most candidates for true elementary building blocks of Nature are bosons or fermions. Different statistics are associated to both kinds of "particles." Therefore, the question: "Are bosons or are fermions the true elementary building blocks?" leads to a hierarchy of elementarity between the two main quantum statistics: Bose–Einstein and Fermi–Dirac.

The main purpose of our paper is further research on the existence of that hierarchy, and on the identification of that statistics which is more elementary. As a by-product, we will obtain more general results valid for other associative algebras than Bose or Fermi ones.

We shall defend the heterodox point of view that it is highly probable that the true elementary building blocks, if they exist, are bosons. This goes against the widespread belief that since tensors can be constructed in terms of spinors, it is impossible that Universe is built from bosons only. The basis of that widespread belief will also be analyzed.

2. CAUTIONS

It is possible that the notion of elementary building blocks of Nature makes no sense; the sequence of macroscopic bodies, molecules, atoms, "elementary" particles,... may be an endless succession. Therefore, unless explicitly stated to the contrary, we shall use, as a working hypothesis, the *assumption* that true elementary building blocks (not necessarily "particles") exist.

There is no direct evidence that Bose and Fermi are the only quantum statistics allowed by Nature [see Section 2.2 of (Kálnay, 1977a)]. For example, in most cases statistics is inferred from the spin-statistics theorem and the measurement of spin. But the spin-statistics theorem only insures that half-integer (or integer) spin "particles" are fermions (respectively, bosons) *if* no other quantum statistics than Bose and Fermi exist, so that such kind of inference is not conclusive. Bose-like (parabose) and Fermi-like (parafermi) statistics are consistent with quantum field theory (Green, 1953). In spite of the chance we think that one should offer to such generalized statistics, we shall *assume* in *this* paper that Bose and Fermi are the only candidates for the quantum statistics of the elementary building blocks. However, much of the research done on the bosonization of parastatistics (Kademova, 1970) makes plausible without proving that our conclusions will not be altered too much if elementary para-"particles" are accepted.

Let us assume for a moment that quarks really exist in Nature (i.e., that they do not simply offer a model to describe it). If this is so, we are certainly not claiming that quarks are bosons! In fact the idea that quarks are not elementary is continuously gaining support, as the current literature on preons (see, e.g., Sikive, 1981) shows. Our defense of bosons as the probable building blocks of Nature will stand for the true elementary building blocks (whose existence we assumed as a working hypothesis), not to specific candidates to fill in that role: atoms were thought to be elementary, one learned they are not; later on protons (etc.) were thought to be elementary and now one thinks they are not; and so on.

It is conceivable that even if true elementary building blocks exist, a suitable selection of them could be such that they are all bosons, but that another selection also exists where they are only a set of fermions. (Also intermediate selections are conceivable.) Now, let us assume that there is no hierarchy of elementarity among both selections. (Hypotheses related to this were considered in the literature.) If this is true, then the controversy on the hierarchy of elementarity between bosons and fermions would be methodological, not physical. Our research will lead to the result that such "democracy" among fermions and bosons is false, at least under the hypothesis of our research.

3. BOSONIZATION: A RÉSUMÉ

The *Bose description of fermions* or, in short, the *bosonization* of fermions, mathematically means a realization of Fermi algebra in terms of elements of Bose algebra. Physically it means that, given a physical theory whose elementary building blocks are fermions (or fermions and bosons), a physically equivalent theory is shown whose elementary building blocks are bosons only. Several different bosonizations are known as, for example, the one by Okubo (1974) and those quoted in Kademova (1970) and Kálnay *et al.* (1973). We shall only be concerned with that bosonization developed in a line of research which began in a paper by Kademova (1970) and was later developed in Kademova and Kálnay (1970), Kálnay (1975; 1977a, b), Kálnay *et al.* (1973), Kálnay and Kademova (1975a, b) and Kálnay and Mac-Cotrina (1976). The simplest way to show this bosonization is to exhibit the *model system* described in (Kálnay, 1977b): One starts with a system of two Fermi-coupled oscillators described in terms of annihilation and creation operators f_i, f_i^+,

$$[f_i, f_j^+]_+ = \delta_{ij} I, \qquad [f_i, f_j]_+ = 0 \tag{1}$$

$i, j = 1,2$, and with the time evolution ruled by a Hamiltonian

$$H_f(f, f^+) \equiv \omega_1 f_1^+ f_1 + \omega_2 f_2^+ f_2 + \lambda H'(f, f^+) \tag{2}$$

λ being a coupling constant. The elementary building blocks of the theory are the Fermi "particles" created by f_1^+ and f_2^+. The bosonization of this model system is achieved by replacing those building blocks by new ones: four Bose oscillators described in terms of annihilation and creation operators b_r, b_r^+,

$$[b_r, b_s^+]_- = \delta_{rs} I, \qquad [b_r, b_s]_- = 0$$
$$r, s = 1,2,3,4 \tag{3}$$

Let $|0\rangle^{\mathfrak{B}}$ be the Bose vacuum,

$$b_r |0\rangle^{\mathfrak{B}} = 0, \qquad \forall r \tag{4}$$

$\mathfrak{B}$, the Bose state vector space, and $\mathfrak{B}_p$, the p-boson subspace. Then one looks for suitable functions $f_i(b, b^+)$ $(i = 1,2)$ of the Bose operators and a well-selected Bose subspace $\mathfrak{B}'$, such that, when $f_i(b, b^+)$ and $f_i^+(b, b^+)$ act on $\mathfrak{B}'$, the Fermi anticommutation rules (1) are retrieved.

The Fermi Hamiltonian (2) can also be expressed in Bose terms, as we shall show later on. In the quoted model the functions $f_i(b, b^+)$ are selected as

$$f_1(b, b^+) \equiv b_1^+ b_2 + b_3^+ b_4, \qquad f_2(b, b^+) \equiv b_1^+ b_3 - b_2^+ b_4 \tag{5a}$$

so that

$$f_1^+(b, b^+) \equiv [f_1(b, b^+)]^+ = b_2^+ b_1 + b_4^+ b_3, f_2^+(b, b^+) = b_3^+ b_1 - b_4^+ b_2 \tag{5b}$$

and $\mathfrak{B}'$ is the one-boson subspace, $\mathfrak{B}' = \mathfrak{B}_1$, span by $b_r^+ |0\rangle^{\mathfrak{B}}$, $r = 1, \ldots, 4$. The reader can easily verify that

$$[f_i(b, b^+), f_j^+(b, b^+)]_+ (b_r^+ |0\rangle^{\mathfrak{B}}) = \delta_{ij} (b_r^+ |0\rangle^{\mathfrak{B}}) \tag{6a}$$

$$[f_i(b, b^+), f_j(b, b^+)]_+ (b_r^+ |0\rangle^{\mathfrak{B}}) = 0, \qquad \begin{matrix} i, j = 1,2 \\ r = 1,2,3,4 \end{matrix} \tag{6b}$$

so that the anticommutation relations (1) of the Fermi algebra are satisfied in $\mathfrak{B}_1$: the one-boson subspace $\mathfrak{B}' = \mathfrak{B}_1$ is the Bose representative of the Fermi state vector space $\mathfrak{F}$. Moreover, by defining

$$|0\rangle^{\mathfrak{F}} \equiv b_1^+ |0\rangle^{\mathfrak{B}} \in \mathfrak{B}_1 \tag{7a}$$

one finds that

$$f_i(b, b^+)|0\rangle^{\mathfrak{F}} = 0, \qquad i = 1,2 \tag{7b}$$

The one-boson state $|0\rangle^{\mathfrak{F}}$ is the representative of the Fermi vacuum. The Bose representatives of the other Fermi states are also easily shown to be one-boson states; in fact, the states

$$f_1^+(b, b^+)|0\rangle^{\mathfrak{F}} \equiv b_2^+ |0\rangle^{\mathfrak{B}} \in \mathfrak{B}_1, f_2^+(b, b^+)|0\rangle^{\mathfrak{F}} \equiv b_3^+ |0\rangle^{\mathfrak{B}} \in \mathfrak{B}_1$$

and

$$f_1^+(b, b^+) f_2^+(b, b^+)|0\rangle^{\mathfrak{F}} = -f_2^+(b, b^+) f_1^+(b, b^+)|0\rangle^{\mathfrak{F}} \equiv b_4^+ |0\rangle^{\mathfrak{B}} \in \mathfrak{B}_1 \tag{8}$$

form, together with (7a) a suitable basis. Notice the explicit antisymmetry shown in equation (8). And also that the Pauli principle is verified since

$$[f_1^+(b, b^+)]^2 (b_r^+ |0\rangle^{\mathfrak{B}}) = [f_2^+(b, b^+)]^2 (b_r^+ |0\rangle^{\mathfrak{B}}) = 0, \qquad r = 1,2,3,4 \tag{9}$$

This is trivial since the $f_i(b, b^+)$ were selected in such a way that equations (6) are satisfied. In mathematical terms, what we have obtained, was a $*$-isomorphism (Rickart, 1960) between the Fermi algebra and its bosonization. Going back to physics, it can be shown (Kálnay and Kademova, 1975a, and Remarks 2.1.1 and 2.1.2 of Kálnay, 1977a) that generally one should distinguish between the Bose Hamiltonian $H^{\mathfrak{B}}(b, b^+)$ which rules the time evolution of the underlying Bose system, and the Bose representative $H_{\mathfrak{B}}(b, b^+)$ of the Fermi Hamiltonian which rules the time evolution of the f_i and f_i^+. However, both operators can be identified, when there is no physical interest to do otherwise, particularly this is the case with the reviewed model: $H_{\mathfrak{B}} = H^{\mathfrak{B}}$. To be more specific, when the coupling $\lambda H'$ is that of the model, it can be shown that the underlying Bose system evolves as a set of uncoupled Bose oscillators of frequencies 0, ω_1, ω_2 and $\omega_1 + \omega_2 - \lambda$. The operator $H_{\mathfrak{B}}$ is

$$H_{\mathfrak{B}}(b, b^+) \equiv H_{\mathfrak{F}}[f(b, b^+), f^+(b, b^+)] \tag{10}$$

and, as a corollary of the above-mentioned isomorphism it has the same spectrum as that of the original Fermi Hamiltonian $H_{\mathfrak{F}}$ [equation (2)]. As a consequence of equation (10) the action of $H_{\mathfrak{B}}(b, b^+)$ in $\mathfrak{B}_1$ is identical to the one of $H_{\mathfrak{F}}(f, f^+)$ on the Fermi state vector space $\mathfrak{F}$ of the original theory. All these statements can be independently checked by direct computation.

When the operators $f_i(b, b^+)$, $f_i^+(b, b^+)$ defined by equations (5) act on the p-bosons subspace $\mathfrak{B}_p$, a Bose description (bosonization) of all the irreducible representations of parafermi algebra (Green, 1953) is obtained, as can also be directly verified. From the mathematical point of view this shows that the bosonization has a power for unifying separate mathematical structures (Kademova, 1970). From the physical side, the same fact opens conjectural but suggestive perspectives for high energy physics (Kálnay, 1977a).

The bosonization of general finite second quantized Fermi systems and the one of quantum Fermi fields follow the same pattern shown in the above simple model so that we shall not enter into details: we shall only summarize the main results (Kademova and Kálnay, 1970; Kálnay *et al.*, 1973; Kálnay, 1975; Kálnay and Mac-Cotrina, 1976). For finite systems with n annihilation and creation operators f_i, f_i^+ of the Fermi system, 2^n annihilation and creation operators b_r, b_r^+ of the Bose system are needed and c-number coefficients F_{irs} can always be found such that

$$f_i(b, b^+) = \sum_{rs} F_{irs} b_r^+ b_s \tag{11}$$

and its Hermitian conjugate satisfy in $\mathfrak{B}_1$ the Fermi algebra. General formulas for the F_{irs} are somewhat complicated, but in all specific cases (i.e., for each fixed numerical value of n) the F_{irs} take values so simple as the ones which for $n = 2$ lead to equation (5).

When a quantum Fermi field $f_\xi(z)$ is bosonized, the c-number F_{irs} are generalized to c-number functions $F_{\xi\zeta\zeta'}(z, x, x')$, where ξ is a spinorial index and ζ, ζ' are tensor indices. In all cases these functions exist, and are such that,

$$f_\xi(z) = \sum_{\zeta\zeta'} \int_\sigma d\sigma \int_\sigma d\sigma' F_{\xi\zeta\zeta'}(z, x, x') b_\zeta^+(x) b_{\zeta'}(x') \tag{12}$$

and its Hermitian conjugate, satisfy the anticommutation relations of quantum Fermi fields. Here σ is a spacelike hypersurface.

We recalled in Section 1 that the generally accepted statement that, while tensors can be constructed in terms of spinors the opposite construction is impossible, was used to preclude the conception of a Universe built from bosons only. One would not forget, however, that Whittaker (1936) has shown that the calculus of *relativistic* spinors is included in the calculus of tensors, as well as related results by Ruse (1936) and Penney (1965a). The latter obtained a tensorial description of neutrinos. In our case, without having to refer to these papers, the manifest convariance of equation (12) shows that bosons can be the only building blocks even if the calculus of relativistic spinors is not included in the calculus of tensors: the tensor indices in the right-hand side of equation (12) are contracted so that only the spinor index remains.

As a final remark, we mention that the Fermi algebra is not the only associative algebra that can be bosonized. Among several other examples in the literature, we can quote that Schwinger (1965) has bosonized the angular momentum algebra. Recently it was shown that all finite associative algebras can be bosonized, a result from which physical and mathematical applications follow; for example, from the bosonization of Dirac γ^μ matrices, a unification of the higher spin relativistic equations of Bargmann–Wigner (1948) and related formalisms is achieved, in the form of a single wave equation (González-Bernardo *et al.* 1982; to be submitted). Our bosonization of an algebra is similar to the well-known bosonization of Lie-Algebras (Schwinger, 1965) and references quoted in (Biedenharn, 1982) and (González-Bernardo *et al.*, to be submitted).

4. FERMIONIZATION AND ITS DIFFICULTIES: SYNTHESIS OF PENNEY'S RESULTS

By analogy with the now popular word "bosonization," let us call *fermionization* of an algebra a realization of that algebra in terms of Fermi annihilation and creation operators. If $\mathfrak{F}$ is the Fermi state vector space, one may consider the eventual possibility of obtaining a fermionization in the whole $\mathfrak{F}$ or in a subspace $\mathfrak{F}' \subset \mathfrak{F}$ only.

Penney (1965b) is concerned with the problem of "making bosons from fermions," i.e., to the eventual fermionization of the Bose algebra. His main result can be expressed, in our terminology, as follows:

> *Theorem 4.1* (Penney). As long as one considers only a finite number of fermions and one imposes $\mathfrak{F}' = \mathfrak{F}$, the fermionization of the Bose algebra is not possible.

His proof is very elegant and uses properties of Clifford algebra. He stresses that the theorem is limited by two hypotheses: (i) finiteness and

(ii) $\mathscr{F}' = \mathscr{F}$. He shows that some fermionizations obtained in the literature are possible just because at least one of these conditions is violated.

5. FERMIONIZATION AND ITS DIFFICULTIES: OUR APPROACH

We shall extend Penney's Theorem 4.1 in two forms: We shall remove the condition $\mathscr{F}' = \mathscr{F}$ and we shall also obtain some results for algebras more general than Bose algebra.

Our building blocks will be fermions, and we shall only consider (as in Penney's theorem) the case in which the number of annihilation and creation operators f_i and f_i^+ is finite: $i = 1, 2, \ldots, n$. Because of the anticommutation relations (1), the more general product $\prod f_i$ equals, up to reordering, $f_1, f_2, \ldots, f_n$. Therefore, the more general function $g(f, f^+)$ of the Fermi operators is of the form

$$g(f, f^+) = \sum_{\substack{K, L = 1 \\ i_1 < i_2 < \cdots < i_K \\ j_1 < j_2 < \cdots < j_L}}^{n} c_{i_1 \cdots i_K j_1 \cdots j_L} f_{i_1}^+ \cdots f_{i_K}^+ f_{j_1} \cdots f_{j_L} \tag{13}$$

where the $c_{\ldots}$ are complex numbers (or, more generally, elements of a field). In equation (13) the operators are ordered in normal form, since from equation (1) it follows that if they were not, they could always be reordered to that form (without changing g) by means of a redefinition of the coefficients $c_{\ldots}$.

Let us now consider an associative algebra $\mathscr{A}$ over a field Φ, with generators A_t. (For example, $\mathscr{A}$ could be the Bose algebra, in which case Φ is the field of the complex numbers and the A_t are the b_r, b_s^+.) We want a fermionization of $\mathscr{A}$, done in terms of the above-mentioned finite number of fermions. The concept of fermionization of the algebra $\mathscr{A}$ will be taken in a similar way as the one of bosonization of an algebra by simply replacing bosons by fermions as elementary building blocks, but it is more general in a certain sense to be discussed below.

We define functions $A_t(f, f^+)$ of the Fermi operators, and select a subspace $\mathscr{F}'$ of the state vector space $\mathscr{F}$ of the fermions such that the operators $A_t(f, f^+)$ act in $\mathscr{F}'$ in closed form:

$$A_t(f, f^+)|\Psi\rangle \in \mathscr{F}', \qquad \forall |\Psi\rangle \in \mathscr{F}', \forall t \tag{14}$$

Let us call $\mathscr{A}_{\mathscr{F}}$ the algebra generated by the $A_t(f, f^+)$. We shall say that $\mathscr{A}_{\mathscr{F}}$ is a fermionization of the original algebra $\mathscr{A}$ iff $\mathscr{A}_{\mathscr{F}}$ is isomorphic to $\mathscr{A}$. (In case

$\mathcal{A}$ is a $*$-algebra (Rickart, 1960) we shall further require that the isomorphism is a $*$-isomorphism.)

Example: If the original algebra is the Bose algebra defined by the commutation relations like (3), with $r, s = 1, 2, \ldots, m$, then $A_t(f, f^+)$ will be a set of functions $b_r(f, f^+)$, $b_s^+(f, f^+)$ such that

$$b_r(f, f^+) \leftrightarrow b_r, \qquad b_s^+(f, f^+) \leftrightarrow b_s^+ \tag{15a}$$

$$b_r(f, f^+)|\Psi\rangle \in \mathcal{F}', \qquad b_s^+(f, f^+)|\Psi\rangle \in \mathcal{F}', \forall|\Psi\rangle \in \mathcal{F}' \tag{15b}$$

$$[b_r(f, f^+), b_s^+(f, f^+)]_-|\Psi\rangle = \delta_{rs}|\Psi\rangle, [b_r(f, f^+), b_s(f, f^+)]_-|\Psi\rangle = 0,$$
$$\forall|\Psi\rangle \in \mathcal{F}' \tag{15c}$$

and that

$$b_s^+(f, f^+) = [b_s(f, f^+)]^+, \qquad \forall r, s \tag{15d}$$

Condition (c) will guarantee that the correspondence (a) is an isomorphism; condition (d) will provide that it is a $*$-isomorphism, i.e., that the involution is preserved; the involution being in this case the familiar Hermitian conjugation. If the functions $b_r(f, f^+)$, $b_s^+(f, f^+)$ exist, a fermionization of the Bose algebra would be achieved.

Turning back to the general case of a general associative algebra $\mathcal{A}$, the fact that the more general function is of the form (13), implies that if the fermionization is possible, the $A_t(f, f^+)$ are polynomials as

$$A_t(f, f^+) = \sum_{\substack{K, L = 1 \\ i_1 < \cdots < i_K \\ j_1 < \cdots < j_L}}^{n} c^t_{i_1 \cdots i_K j_1 \cdots j_L} f^+_{i_1} \cdots f^+_{i_K} f_{j_1} \cdots f_{j_L}, \tag{16}$$

where the coefficients $c^t_{\ldots}$ belong to Φ. (They are complex numbers in the example.)

Now we are prepared to prove our theorems.

> *Theorem 5.1.* Let us assume that an associative algebra $\mathcal{A}$ allows a fermionization in terms of a finite number of fermions. Then a matrix representation of $\mathcal{A}$ exists.

Proof. It is known that for finite number of fermions the operators f_i, f_i^+ allow a finite matrix irreducible representation, its explicit form given (up to unitary transformations) by Jordan and Wigner (1928). The projection operator Λ such that

$$\Lambda\mathfrak{F} = \mathfrak{F}', \qquad \Lambda^2 = \Lambda \tag{17}$$

as well as the polynomials $A_t(f, f^+)$, are functions of f, f^+, belonging to the same matrix representation. Therefore, a finite matrix representation of the algebra $\mathcal{Q}_{\mathfrak{F}}$ generated by the $A_t(f, f^+)$ exists, such that the $A_t(f, f^+)$ act in closed form in $\mathfrak{F}'$. Since the possibility of fermionization of $\mathcal{Q}$ implies that $\mathcal{Q}_{\mathfrak{F}}$ is isomorphic to $\mathcal{Q}$, it results that a finite matrix representation of $\mathcal{Q}$ also exists. ■

Theorem 5.2. As long as one considers only a finite number of fermions, the fermionization of an irreducible representation of a finite Bose algebra is not possible.

Proof. Put $\mathcal{Q}$ = the Bose algebra defined by equations (3) (but now with $r, s = 1, 2, \ldots, m$) or, more rigorously, by its Weyl form (Wightman and Schweber, 1955), and assume that the fermionization exists. Irreducibility implies, up to a unitary transformation, the Schrödinger representation (Wightman and Schweber, 1955; Weyl, 1931)

$$b_r = (q_r + \partial/\partial q_r)/\sqrt{2} \tag{18}$$

which does not allow for a finite matrix representation, contradicting Theorem 5.1. ■

Therefore, we obtained a generalization of Theorem 4.1 since condition $\mathfrak{F}' \neq \mathfrak{F}$ was not used, and the proof is even simpler.

A theorem like 5.2 could be also stated, without changes, for a Fock representation of an infinite Bose algebra, but we are not writing it since it would look artificial to try to represent such infinite algebra in terms of a *finite* Fermi algebra.

The condition that the operators act in closed form in $\mathfrak{F}'$ is essential for Theorems 5.1 and 5.2. This can be visualized in terms of an example exhibited by Penney (1965b) in his equations (21)–(24). Penney's purpose was to show that apparent contradictions to this theorem were not true in this way. Penney's example can be discussed as follows: consider a Fermi system with no particle state $|0\rangle^{\mathfrak{F}}$ and only one annihilation operator f; then $|0\rangle^{\mathfrak{F}}$ and $f^+|0\rangle^{\mathfrak{F}}$ span the Fermi state vector space $\mathfrak{F}$; take $\mathfrak{F}' = \{|0\rangle\}$; then not only $(ff^+ + f^+f)\mathfrak{F}' = \mathfrak{F}'$ is true, but also $(ff^+ - f^+f)\mathfrak{F}' = \mathfrak{F}'$ so that a fermionization of Bose algebra seems to be obtained in $\mathfrak{F}'$. This fermionization is not possible indeed, but not because $\mathfrak{F}'$ is a proper subspace of $\mathfrak{F}$ but since the operators do not act in closed form in $\mathfrak{F}$, for the reason that $f^+|0>^{\mathfrak{F}} \notin \mathfrak{F}'$.

We mentioned that the fermionization of algebras under discussion is more general than the corresponding bosonization. In fact, the equation analogous to (16) for a bosonization of an algebra would be

$$A_t(b, b^+) = \sum_{\substack{K, L=1 \\ r_1 < \cdots < r_K \\ s_1 < \cdots < s_L}}^{m} c^t_{r_1 \cdots r_K s_1 \cdots s_L} b^+_{r_1} \cdots b^+_{r_K} b_{s_1} \cdots b_{s_L} \tag{19}$$

In the case of bosonization we have proven (Kademova, 1970; González-Bernardo *et al.*, 1982; to be submitted) that it can be achieved with $K = L = 1$. In other words, it is not necessary to study a general operator like (19) in order to have the bosonization of an algebra. In particular, for $K = L = 1$ it is obvious that the operator (19) acts in closed form in $\mathfrak{B}' = \mathfrak{B}_1$ (Section 3) so that for the case of a bosonization the analog of equation (14) is automatically satisfied and does not need to be assumed. On the other hand, if we would start from the analysis of the possibility of fermionization by only considering $K = L = 1$ in equation (16), the doubt would arise that although that case is not possible, a more general fermionization could still make sense.

Finally, let us quote that certain algebras allow for a known fermionization such that $K = L = 1$ in equation (16). Let us quote, e.g., (i) the well-known fermionization of the spin-1/2 algebra and (ii) the fermionization by Jagannathan and Vasudevan (1978) of para-Grassmann (Kálnay, 1976) algebras. On the other hand, Jagannathan and Vasudevan (1978) fermionization of the parafermi (Green, 1953) algebra falls into the case (16) with $K, L > 1$ and $\mathfrak{F}' = \mathfrak{F}$.

6. DISCUSSION

Let us assume that (i) true elementary building blocks exist, that (ii) they can only be bosons or fermions obeying (at least at certain stage of the construction of the formalism) the canonical commutation–anticommutation relations, and that (iii) the index which labels the creation operators of the elementary building blocks only takes a finite number of values. It follows from the papers summarized in Section 3 that this set of assumptions is consistent if the elementary building blocks are bosons, since an explicit bosonization was constructed. And from Theorem 5.2 it results that (i), (ii), and (iii) are inconsistent if the building blocks are fermions. Therefore, *as long our assumptions are correct, a hierarchy of elementarity* *exists among the quantum statistics*: Bose–Einstein statistics stands for objects which can be elementary; Fermi–Dirac cannot stand.

If (i) is false because an infinite sequence of sets of building blocks exist, each term of the sequence serving to describe the former terms but not vice versa (Section 2), then our conclusion is still correct in the sense that at each stage of the sequence, bosonization (but not fermionization) is possible.

If, on the other hand, hypothesis (ii) and/or (iii) are not correct, then we have not rigorously proven that a Bose-privileged hierarchy of elementarity exists among the quantum statistics, although they suggest that such privileged hierarchy is likely.

In the case that assumption (iii) fails, it seems that a completely different kind of analysis should be used to search for that hierarchy, as the following intuitive reasoning suggests: One could say that for the *finite* case our result that fermions can always be bosonized but that bosons can never be fermionized is a consequence of the following fact: If (iii) holds, then the dimension of Fermi Fock space is finite (Pauli Principle) while that of Bose Fock space is infinite. (This argument is only an intuitive one: the possibility of bosonization of fermions is suggested but not proven.) However, if (iii) fails, at least one half of the research is done: we reviewed in Section 3 that bosonization of Fermi quantum fields in terms of Bose quantum fields holds. What remains to be investigated is the eventual fermionization of Bose fields.

Bose and Fermi algebras are not the only ones used in physics. Some of them can be fermionized. But all finite algebras can be bosonized (González-Bernardo *et al.*, 1982; to be submitted). Therefore, bosons are elementary even in other sense: all physical relations expressed in terms of a finite algebra can be reworded in pure Bose terms. This is true even if the finite algebra is related to an infinite system: for example, the bosonization of Dirac γ^μ matrices (González-Bernardo *et al.*, 1982; to be submitted) reviewed in Section 3.

The greatest risk of misunderstandings in this kind of work comes from the phrase "from spinors you can construct tensors but not vice versa, so that fermionization is possible, bosonization impossible." We discussed the matter in Section 3, but the point is so crucial that we touch it again: bosons and fermions "are" not only tensors and (respectively) spinors. Bosons "are" tensors plus commutation rules, fermions "are" spinors plus anticommutation rules. When the commutation (respectively, anticommutation) rules are taken into account, it results that, at least if (i)–(iii) hold, "bosonization is possible, fermionization impossible." And for the above spinor construction problem, we recall that the manifest covariance of equation (12) shows that the Fermi fields we built in terms of Bose quantum fields are spinors, while the latter are tensors.

And God said: "Let there be light."
And God saw that light was good.
(Genesis, 1.3, 1.4)

We like the beauty of light, we like the beauty of photons.

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